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# *On the Problem of Two Fixed Centres and Certain of Its Generalizations.*

BY ADAM MILLER HILTEBEITEL.

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The classic problem of two fixed centres may be stated thus: To determine the motion of a body attracted by two fixed Newtonian centres of force. It was first solved by Euler and since Euler generalizations of the problem have been made. In this paper are given the resolution of the most general two-centre problem of which the variables are separable and a discussion of the trajectories of a less general form of the problem.

PROBLEM: To study the motion of a material body $M$ acted upon by five centres of force, $K_1$, $K_2$, $K_3$, $K_4$, $K_5$, which are the two real foci, the centre, and the two imaginary foci, respectively, of a system of confocal conics, by forces

$$R_1 = -m r_1 - \frac{m_1}{r_1^2}, \quad R_2 = -m r_2 - \frac{m_2}{r_2^2}, \quad R_3 = -m_3 r_3,$$

$$R_4 = -m' r_4 - \frac{m_4 + i m_5}{r_4^2}, \quad R_5 = -m' r_5 - \frac{m_4 - i m_5}{r_5^2},$$

respectively, and by an additional force always parallel to the line $K_4 K_5$ and varying inversely as the cube of the distance of the moving body $M$ from the fixed plane perpendicular to $K_4 K_5$ at $K_3$, the centres $K_3$, $K_4$, $K_5$ being fixed, while the real foci $K_1$, $K_2$ are always in the meridian plane $K_4 M K_5$.

Let $x$, $y_1$, $z_1$ be the coordinates of the moving body $M$ referred to a fixed system of rectangular axes whose origin is at $K_3$ and whose $x$-axis coincides with the line passing through the imaginary foci $K_4$, $K_5$, as in Fig. 1. Imagine a plane through the $x$-axis and the moving body $M$, and let the intersection of this moving plane with the fixed $y_1 z_1$-plane be $YY'$. This line will always lie in the meridian plane $K_4 M K_5$ and, as this plane rotates about the $x$-axis, $YY'$ will rotate about the origin. Every point in this line will describe a circle in the $y_1 z_1$-plane, whose centre is $K_3$, and upon it the two real foci are taken. Of the five centres of force, therefore, $K_3$, $K_4$, $K_5$ are fixed, while the other two move on a circle whose centre is $K_3$.

The motion of the meridian plane, and therefore that of the centres $K_1$, $K_2$, is due to that element of the initial velocity of $M$ which is perpendicular to the initial position of the meridian plane. But this element of the initial velocity not only causes the rotation of the meridian plane, but also influences the nature of the path described by the moving body in this plane. By the aid of the principle of areas the motion of $M$ in the rotating plane can be completely separated from the motion of rotation about the $x$-axis, as Jacobi* has done for the simpler problem.

The perpendicular distances of $M$ from the $x$-axis and the line $YY'$ are designated by $y$ and $x$ respectively. Regarding $XX'$ and $YY'$ as a system of rectangular axes in the meridian plane, then $x$, $y$ are the rectangular coordinates of $M$ defining its position in this plane. If at the beginning of motion the plane $K_4MK_5$ coincides with the fixed plane $XK_3Y_1$, and during the time $t$ describes the angle $\phi$, as indicated in Fig. 1, then

$$y_1 = y\cos\phi, \qquad z_1 = y\sin\phi, \qquad y_1^2 + z_1^2 = y^2. \tag{1}$$

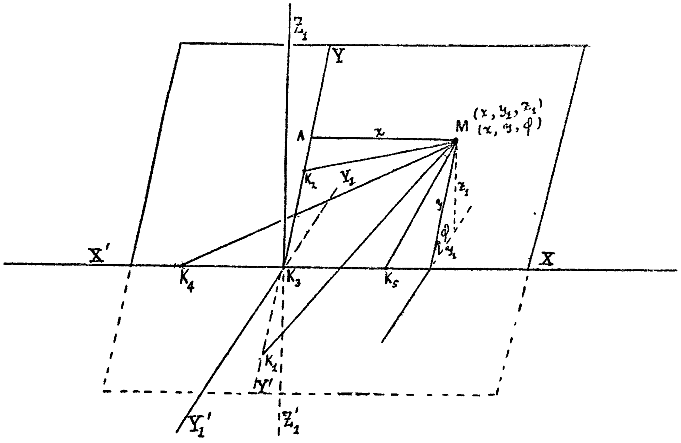

FIG. 1.

Let $2c$ denote the distance $K_1K_2$ and $2ic$ the distance $K_4K_5$. Let the rectangular coordinates of $K_1$, $K_2$ in the $Y_1Z_1$-plane be $(-a, -b)$, $(a, b)$, respectively, and the distances $K_1M$, $K_2M$, $K_3M$, $K_4M$, $K_5M$ be respectively $r_1$, $r_2$, $r_3$, $r_4$, $r_5$. We now have:

$$a^2 + b^2 = c^2, \quad by_1 - az_1 = 0, \tag{2}$$

$$r_1^2 = x^2 + (y_1 + a)^2 + (z_1 + b)^2, \quad r_2^2 = x^2 + (y_1 - a)^2 + (z_1 - b)^2, \tag{3}$$

$$r_3^2 = x^2 + y_1^2 + z_1^2, \quad r_4^2 = (x + ic)^2 + y_1^2 + z_1^2, \quad r_5^2 = (x - ic)^2 + y_1^2 + z_1^2. \tag{4}$$

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* *Vorlesungen über Dynamik*, No. 29. *Werke*, Supplement, pp. 221–231.

The form of the forces $R_1$, $R_2$, $R_4$, $R_5$ is accounted for by supposing that there are two masses at each of the corresponding centres, one acting inversely as the square of the distance and the other directly as the distance. At $K_1$, $K_2$ the masses acting inversely as the squares of the distances, denoted by $m_1$, $m_2$ respectively, may be any real quantities whatever, while the masses acting directly as the distances must be real and equal to each other. These latter masses are denoted by $m$. But at $K_4$, $K_5$ the masses acting inversely as the squares of the distances will be denoted by the conjugate complex quantities, $m_4 + i m_5$, $m_4 - i m_5$, $i$ being the imaginary unit, and the masses acting as the direct distances may be any real quantity, as $m'$. Under these hypotheses as to the masses, the potential function will be real and the separation of the variables in the Hamilton-Jacobi partial differential equation can be effected by the aid of elliptic coordinates.

This problem includes, as particular cases, all the various forms of the two-centre problem. The form of the potential function given by Darboux* may be obtained by setting $m = m' = 0$, or by compounding the two forces $-m r_1$, $-m r_2$ into the single force $-2 m r_3$ and suppressing the forces $-m' r_4$, $-m' r_5$.

These four forces together with the force $-m_3 r_3$ give rise to the following terms in the potential function:

$$- \tfrac{1}{2} m r_1^2 - \tfrac{1}{2} m r_2^2 - \tfrac{1}{2} m_3 r_3^2 - \tfrac{1}{2} m' r_4^2 - \tfrac{1}{2} m' r_5^2;$$

and it can easily be shown that this expression is equivalent to

$$- (m + \tfrac{1}{2} m_3 + m') r_3^2 - c^2 (m - m').$$

Hence, even on retaining these forces, the potential function of this problem differs from that of Darboux's problem only by a constant term. Velde's† problem is obtained from the present problem by omitting the forces $R_4$, $R_5$ as well as that parallel to the $x$-axis, and by regarding the motion as in a fixed plane through the centres $K_1$, $K_2$, which are also supposed fixed. By simply omitting the forces $R_4$, $R_5$, we have the most general form of the two-centre problem as formulated by Liouville.* Retaining only the forces $R_1$, $R_2$ and regarding the corresponding centres as fixed, we have the problem as extended by Lagrange;† and if, in addition, we suppress the terms $-mr_1$, $-mr_2$, in $R_1$, $R_2$ respectively, we have the original two-centre problem of Euler.‡

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* *Archives Néerlandaises des Sciences*, Ser. 2, Vol. VI, pp. 371-376.

† *Programm der ersten höheren Bürgerschule zu Berlin*, 1889.

Differentiating equations (2), (3), (4), we have

$$a\,da + b\,db = 0, \quad b\,dy_1 + y_1\,db - a\,dz_1 - z_1\,da = 0, \tag{5}$$

$$\left.\begin{aligned} r_1\,dr_1 &= x\,dx + (y_1 + a)\,dy_1 + (y_1 + a)\,da + (z_1 + b)\,dz_1 + (z_1 + b)\,db, \\ r_2\,dr_2 &= x\,dx + (y_1 - a)\,dy_1 - (y_1 - a)\,da + (z_1 - b)\,dz_1 - (z_1 - b)\,db, \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} r_3\,dr_3 &= x\,dx + y_1\,dy_1 + z_1\,dz_1, \\ r_4\,dr_4 &= (x + ic)\,dx + y_1\,dy_1 + z_1\,dz_1, \\ r_5\,dr_5 &= (x - ic)\,dx + y_1\,dy_1 + z_1\,dz_1. \end{aligned}\right\} \tag{7}$$

But

$$(y_1 + a)\,da + (z_1 + b)\,db = \frac{z_1}{b}(ada + bdb) + ada + bdb = 0,$$

$$-(y_1 - a)\,da - (z_1 - b)\,db = -\frac{z_1}{b}(ada + bdb) + ada + bdb = 0;$$

and therefore the equations (6) become

$$\left.\begin{aligned} r_1\,dr_1 &= xdx + (y_1 + a)\,dy_1 + (z_1 + b)\,dz_1, \\ r_2\,dr_2 &= xdx + (y_1 - a)\,dy_1 + (z_1 - b)\,dz_1. \end{aligned}\right\} \tag{8}$$

It will now be necessary to express the distances in terms of the variables $x$, $y$. This can be done without difficulty by aid of the relations (1) and (2), and we have

$$\left.\begin{aligned} r_1^2 = x^2 + (y + c)^2, \quad r_2^2 = x^2 + (y - c)^2, \quad r_3^2 = x^2 + y^2, \\ r_4^2 = (x + ic)^2 + y^2, \quad r_5^2 = (x - ic)^2 + y^2. \end{aligned}\right\} \tag{9}$$

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* *Journ. de Math.*, Vol. XI, pp. 345–378; Vol. XII, pp. 410–444; Vol. XIII, pp. 34–37. *Connaissance de Temps pour* 1849, pp. 255–256.

† *Miscellanea Taurinensia*, Vol. IV, pp. 118–243: *Œuvres*, Vol. II, pp. 67–121; *Mécanique Analytique*, 1st Ed., pp. 262–286; 2nd Ed., Vol. II, pp. 108–121; *Œuvres*, Vol. XII, pp. 101–114.

‡ *Nov. Comm. Acad. Imp. Petropolitanae*, Vol. X, pp. 207–242; Vol. XI, pp. 152–184. *Mémoires de l'Acad. de Berlin*, Vol. XI, pp. 228–249.

Taking the mass $M$ as the unit mass, the differential equations of motion may now be written as follows:

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} &= -mx - mx - m_3x - m'(x+ic) - m'(x-ic) - \frac{m_1x}{r_1^3} - \frac{m_2x}{r_2^3} \\
&\quad - \frac{(m_4+im_5)(x+ic)}{r_4^3} - \frac{(m_4-im_5)(x-ic)}{r_5^3} - \frac{\beta}{x^3}, \\
\frac{d^2y_1}{dt^2} &= -m(y_1+a) - m(y_1-a) - m_3y_1 - m'y_1 - m'y_1 \\
&\quad - \frac{m_1(y_1+a)}{r_1^3} - \frac{m_2(y_1-a)}{r_2^3} - \frac{(m_4+im_5)y_1}{r_4^3} - \frac{(m_4-im_5)y_1}{r_5^3}, \\
\frac{d^2z_1}{dt^2} &= -m(z_1+b) - m(z_1-b) - m_3z_1 - m'z_1 - m'z_1 \\
&\quad - \frac{m_1(z_1+b)}{r_1^3} - \frac{m_2(z_1-b)}{r_2^3} - \frac{(m_4+im_5)z_1}{r_4^3} - \frac{(m_4-im_5)z_1}{r_5^3}.
\end{aligned}\right\} \quad (10)$$

Multiplying equations (10) respectively by $dx$, $dy_1$, $dz_1$, adding and integrating, we obtain the vis viva integral

$$\left.\begin{aligned}
\tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2\right\} &= -\tfrac{1}{2}mr_1^2 - \tfrac{1}{2}mr_2^2 - \tfrac{1}{2}m_3r_3^2 - \tfrac{1}{2}m'r_4^2 \\
&\quad - \tfrac{1}{2}m'r_5^2 + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_4+im_5}{r_4} + \frac{m_4-im_5}{r_5} + \tfrac{1}{2}\cdot\frac{\beta}{x^2} + h.
\end{aligned}\right\} \quad (11)$$

Now multiplying the second equation of (10) by $z_1$, and the third by $y_1$, and then subtracting, we obtain

$$\left.\begin{aligned}
y_1\left(\frac{dz_1}{dt}\right)^2 - z_1\left(\frac{dy_1}{dt}\right)^2 &= -m(by_1-az_1) + m(by_1-az_1) \\
&\quad - \frac{m_1(by_1-az_1)}{r_1^3} + \frac{m_2(by_1-az_1)}{r_2^3};
\end{aligned}\right\} \quad (12)$$

and therefore, since $by_1 - az_1 = 0$, we have

$$y_1\frac{d^2z_1}{dt^2} - z_1\frac{d^2y_1}{dt^2} = 0. \quad (13)$$

Integrating this equation, we have

$$y_1\frac{dz_1}{dt} - z_1\frac{dy_1}{dt} = \alpha, \quad (14)$$

where $\alpha$ is the constant of integration. This integral equation shows that the areas described by $y$, as the meridian plane rotates about the $x$-axis, are proportional to the times.

By the aid of (1) we find

$$\left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2 = \left(\frac{dy}{dt}\right)^2 + y^2\left(\frac{d\phi}{dt}\right)^2, \tag{15}$$

$$y_1\frac{dz_1}{dt} - z_1\frac{dy_1}{dt} = y^2\frac{d\phi}{dt}. \tag{16}$$

Substituting these values in (11) and (14), we have

$$\left.\begin{aligned}\tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} &= -\tfrac{1}{2}m r_1^2 - \tfrac{1}{2}m r_2^2 - \tfrac{1}{2}m_3 r_3^2 - \tfrac{1}{2}m' r_4^2 - \tfrac{1}{2}m' r_5^2 \\ &\quad + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_4 + im_5}{r_4} + \frac{m_4 - im_5}{r_5} + \tfrac{1}{2}\cdot\frac{\beta}{x^2} - \tfrac{1}{2}\cdot\frac{\alpha^2}{y^2} + h,\end{aligned}\right\} \tag{17}$$

$$y^2\frac{d\phi}{dt} = \alpha, \tag{18}$$

in which the variables $y_1$, $z_1$, $a$, $b$ no longer appear.

Differentiating the equation $y = \sqrt{y_1^2 + z_1^2}$ twice with regard to $t$, we get

$$\frac{d^2 y}{dt^2} = \frac{z_1\dfrac{d^2 z_1}{dt^2} + y_1\dfrac{d^2 y_1}{dt^2}}{y} + \frac{(y_1^2 + z_1^2)\left\{\left(\dfrac{dy_1}{dt}\right)^2 + \left(\dfrac{dz_1}{dt}\right)^2\right\} - \left(y_1\dfrac{dy_1}{dt} + z_1\dfrac{dz_1}{dt}\right)^2}{y^3}. \tag{19}$$

The second term of the right-hand member of this equation reduces to

$$\frac{1}{y^3}\left(y_1\frac{dz_1}{dt} - z_1\frac{dy_1}{dt}\right)^2 = \frac{\alpha^2}{y^3},$$

and the first term becomes

$$\begin{aligned}&-m(y+c) - m(y-c) - m_3 y - m' y - m' y \\ &\qquad - \frac{m_1(y+c)}{r_1^3} - \frac{m_2(y-c)}{r_2^3} - \frac{(m_4 + im_5)\,y}{r_4^3} - \frac{(m_4 - im_5)\,y}{r_5^3}.\end{aligned}$$

Substituting these values in (19), we find that the original equations of motion may be replaced by the following system:

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -mx - mx - m_3 x - m'(x + ic) - m'(x - ic) \\ &\qquad - \frac{m_1 x}{r_1^3} - \frac{m_2 x}{r_2^3} - \frac{(m_4 + im_5)(x + ic)}{r_4^3} - \frac{(m_4 - im_5)(x - ic)}{r_5^3} - \frac{\beta}{x^3}, \\ \frac{d^2y}{dt^2} &= -m(y+c) - m(y-c) - m_3 y - m' y - m' y \\ &\qquad - \frac{m_1(y+c)}{r_1^3} - \frac{m_2(y-c)}{r_2^3} - \frac{(m_4 + im_5)\,y}{r_4^3} - \frac{(m_4 - im_5)\,y}{r_5^3} + \frac{\alpha^2}{y^3}, \\ \frac{d\phi}{dt} &= \frac{\alpha}{y^2}.\end{aligned}\right\} \tag{20}$$

The first two of these equations involve only the variables $x$ and $y$ and define the position of the moving body in the meridian plane. The third equation gives the motion of rotation about the $x$-axis.

If we set

$$U = -\tfrac{1}{2}\, m r_1^2 - \tfrac{1}{2}\, m r_2^2 - \tfrac{1}{2}\, m_3 r_3^2 - \tfrac{1}{2}\, m' r_4^2 - \tfrac{1}{2}\, m' r_5^2$$
$$+ \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{(m_4 + i m_5)}{r_4} + \frac{(m_4 - i m_5)}{r_5} + \tfrac{1}{2}\frac{\beta}{x^2} - \tfrac{1}{2}\frac{\alpha^2}{y^2},$$

the equations (20) may be written in the form

$$\frac{d^2 x}{dt^2} = \frac{\partial U}{\partial x}, \quad \frac{d^2 y}{dt^2} = \frac{\partial U}{\partial y}, \quad \frac{d\phi}{dt} = -\frac{\partial U}{\partial \alpha}.$$

On introducing elliptic coordinates $\lambda$, $\mu$ defined by the equations

$$\lambda = \tfrac{1}{2}(r_1 + r_2), \qquad \mu = \tfrac{1}{2}(r_1 - r_2), \tag{21}$$

the functions $T$ and $U$ become

$$T = \tfrac{1}{2}(\lambda^2 - \mu^2)\left\{\frac{1}{\lambda^2 - c^2}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{\mu^2 - c^2}\left(\frac{d\mu}{dt}\right)^2\right\},$$

$$U = \frac{1}{\lambda^2 - \mu^2}\Big\{\Big(-(m + \tfrac{1}{2} m_3 + m')\lambda^4 + c^2(\tfrac{1}{2} m_3 + 2m')\lambda^2 + (m_1 + m_2)\lambda$$
$$+ 2 m_4 \sqrt{\lambda^2 - c^2} + \tfrac{1}{2}\frac{\beta c^2}{\lambda^2 - c^2} - \tfrac{1}{2}\frac{\alpha^2 c^2}{\lambda^2}\Big) + \Big((m + \tfrac{1}{2} m_3 + m')\mu^4$$
$$- c^2(\tfrac{1}{2} m_3 + 2m')\mu^2 - (m_1 - m_2)\mu - 2 m_5 \sqrt{c^2 - \mu^2} - \tfrac{1}{2}\frac{\beta c^2}{\mu^2 - c^2} + \tfrac{1}{2}\frac{\alpha^2 c^2}{\mu^2}\Big)\Big\},$$

by virtue of the relations

$$r_1 = \lambda + \mu, \qquad r_2 = \lambda - \mu, \qquad r_3 = \sqrt{\lambda^2 + \mu^2 - c^2},$$
$$r_4 = \sqrt{\lambda^2 - c^2} + \sqrt{\mu^2 - c^2}, \qquad r_5 = \sqrt{\lambda^2 - c^2} - \sqrt{\mu^2 - c^2},$$
$$c^2 y^2 = \lambda^2 \mu^2, \qquad c^2 x^2 = (\lambda^2 - c^2)(c^2 - \mu^2).$$

Both $T$ and $U$ have the form required for the separation of the variables in the Hamilton-Jacobi equation. $U$, however, is an irrational function of the coordinates $\lambda$ and $\mu$, but it is real for all real values of $\lambda > c$ and for all values of $\mu$ numerically less than $c$. It is a real function, therefore, for all values of $\lambda$ and $\mu$ consistent with their definition.

Setting $H = T - U$, the differential equations defining the motion in the meridian plane assume the canonical form

$$\left\{\begin{aligned} \frac{d\lambda}{dt} &= \frac{\partial H}{\partial p_1}, & \frac{dp_1}{dt} &= -\frac{\partial H}{\partial \lambda}, \\ \frac{d\mu}{dt} &= \frac{\partial H}{\partial p_2}, & \frac{dp_2}{dt} &= -\frac{\partial H}{\partial \mu}, \end{aligned}\right.$$

where the conjugate variables $p_1$ and $p_2$ are given by the relations

$$p_1 = \frac{\partial T}{\partial \lambda'}, \quad p_2 = \frac{\partial T}{\partial \mu'}.$$

The Hamilton-Jacobi equation,

$$\tfrac{1}{2}\left\{\frac{\lambda^2 - c^2}{\lambda^2 - \mu^2}\left(\frac{\partial W}{\partial \lambda}\right)^2 - \frac{\mu^2 - c^2}{\lambda^2 - \mu^2}\left(\frac{\partial W}{\partial \mu}\right)^2\right\} = U + h,$$

becomes, on inserting the value of $U$ in full and arranging,

$$\begin{aligned} &\tfrac{1}{2}(\lambda^2 - c^2)\left(\frac{\partial W}{\partial \lambda}\right)^2 - \Big\{-(m + \tfrac{1}{2} m_3 + m')\lambda^4 + (\tfrac{1}{2} m_3 c^2 + 2 m' c^2 + h)\lambda^2 \\ &\qquad + (m_1 + m_2)\lambda + 2 m_4 \sqrt{\lambda^2 - c^2} + \tfrac{1}{2}\frac{\beta c^2}{\lambda^2 - c^2} - \tfrac{1}{2}\frac{\alpha^2 c^2}{\lambda^2}\Big\} \\ &= \tfrac{1}{2}(\mu^2 - c^2)\left(\frac{\partial W}{\partial \mu}\right)^2 + \Big\{(m + \tfrac{1}{2} m_3 + m')\mu^4 - (\tfrac{1}{2} m_3 c^2 + 2 m' c^2 + h)\mu^2 \\ &\qquad - (m_1 - m_2)\mu - 2 m_5 \sqrt{c^2 - \mu^2} - \tfrac{1}{2}\frac{\beta c^2}{\mu^2 - c^2} + \tfrac{1}{2}\frac{\alpha^2 c^2}{\mu^2}\Big\}. \end{aligned}$$

In this equation the variables are separated. Hence by setting each member equal to the same constant and putting for short

$$\begin{aligned} \Lambda &= f(\lambda) + (m_1 + m_2)\lambda^5 + 2 m_4 \lambda^4 \sqrt{\lambda^2 - c^2} - 2 m_4 c^2 \lambda^2 \sqrt{\lambda^2 - c^2} \\ &\qquad - (m_1 + m_2) c^2 \lambda^3, \\ \mathrm{M} &= f(\mu) + (m_1 - m_2)\mu^5 - 2 m_5 \mu^4 \sqrt{c^2 - \mu^2} + 2 m_5 c^2 \mu^2 \sqrt{c^2 - \mu^2} \\ &\qquad - (m_1 - m_2) c^2 \mu^3, \\ f(s) &= -(m + \tfrac{1}{2} m_3 + m') s^8 + (m c^2 + m_3 c^2 + 3 m' c^2 + h) s^6 \\ &\qquad - (\tfrac{1}{2} m_3 c^4 + 2 m' c^4 + h c^2 - k) s^4 - (k + \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta) c^2 s^2 + \tfrac{1}{2}\alpha^2 c^4, \end{aligned}$$

we obtain the two equations

$$\frac{\partial W}{\partial \lambda} = \frac{\sqrt{2}}{\lambda(\lambda^2 - c^2)}\sqrt{\Lambda}, \quad \frac{\partial W}{\partial \mu} = \frac{\sqrt{2}}{\mu(\mu^2 - c^2)}\sqrt{\mathrm{M}} \tag{22}$$

to determine the function $W$.

Multiplying equations (22) by $d\lambda$, $d\mu$ respectively and integrating, we find

$$W(\lambda) = \sqrt{2}\int \frac{d\lambda}{\lambda(\lambda^2 - c^2)}\sqrt{\Lambda}, \quad W(\mu) = \sqrt{2}\int \frac{d\mu}{\mu(\mu^2 - c^2)}\sqrt{\mathrm{M}},$$

$$W = W(\lambda) + W(\mu).$$

Two integrals of the equations of motion then are

$$t + h_1 = \frac{1}{\sqrt{2}}\int \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \frac{1}{\sqrt{2}}\int \frac{\mu^3 d\mu}{\sqrt{\mathrm{M}}}, \tag{23}$$

$$k_1 = \frac{1}{\sqrt{2}}\int \frac{\lambda d\lambda}{\sqrt{\Lambda}} + \frac{1}{\sqrt{2}}\int \frac{\mu d\mu}{\sqrt{\mathrm{M}}} \tag{24}$$

the first of which defines the time corresponding to any position of the moving body in its path and the second the path described in the meridian plane.

The intermediary integrals (22) define the velocities, and they may be written as follows:

$$(\lambda^2 - \mu^2)\frac{d\lambda}{dt} = \frac{\sqrt{2}}{\lambda}\sqrt{\Lambda}, \quad (\lambda^2 - \mu^2)\frac{d\mu}{dt} = -\frac{\sqrt{2}}{\mu}\sqrt{\mathrm{M}}. \tag{25}$$

From (23) and (24), we find by differentiation

$$\sqrt{2}\, dt = \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \frac{\mu^3 d\mu}{\sqrt{\mathrm{M}}}, \tag{26}$$

$$\frac{\lambda d\lambda}{\sqrt{\Lambda}} + \frac{\mu d\mu}{\sqrt{\mathrm{M}}} = 0, \tag{27}$$

and with the aid of (26) and (27) we derive from the third of the equations (20) the relation

$$d\phi = -\frac{\alpha c^2}{\sqrt{2}}\left(\frac{d\lambda}{\lambda\sqrt{\Lambda}} + \frac{d\mu}{\mu\sqrt{\mathrm{M}}}\right), \tag{28}$$

and after integration

$$\phi + \alpha_1 = -\frac{\alpha c^2}{\sqrt{2}}\left\{\int \frac{d\lambda}{\lambda\sqrt{\Lambda}} + \int \frac{d\mu}{\mu\sqrt{\mathrm{M}}}\right\}, \tag{29}$$

which defines $\phi$, the angle of rotation of the meridian plane. With this integral the solution of the problem by quadratures is complete. The three final integrals are (23), (24), (29), and the six arbitrary constants of integration are $h, k, h_1, k_1, \alpha, \alpha_1$, their values depending upon the initial conditions of the motion.

By their definition (21), the coordinates $\lambda$, $\mu$ must always satisfy the inequalities

$$\infty > \lambda \geq c, \qquad -c \geq \mu \geq c,$$

and in order that there may be real motion, they must also satisfy the conditions

$$\Lambda \geq 0, \qquad \mathrm{M} \geq 0,$$

not only at the beginning of motion but also throughout the motion. Further, $\lambda$ can never take the value $+c$ nor approach $+c$ indefinitely, for in this case the potential function $U$ becomes infinite. For the same reason, $\mu$ can never take the values $\pm c$ nor zero, nor approach these values indefinitely. Therefore $\lambda$ must have a minimum value, say $a_1$, greater than $+c$, and $\mu$ must have a maximum and a minimum value in each of the intervals $(0, +c)$, $(-c, 0)$. Let the maximum and minimum values in the interval $(0, +c)$ be respectively $b_1$ and $b_2$, and in the interval $(-c, 0)$, $\beta_1$ and $\beta_2$ respectively. Again, if we suppose all the masses to be essentially positive, then for increasing values of $\lambda$, the function $\Lambda$ finally becomes negative and remains negative. Hence $\lambda$ can not increase indefinitely and the maximum value of $\lambda$ for which $\Lambda$ is positive or zero we shall denote by $a_2$. Hence $\lambda$ and $\mu$ must satisfy the following conditions:

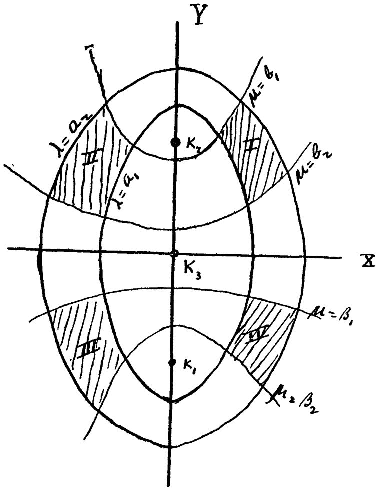

FIG. 2.

$$a_2 \geq \lambda \geq a_1, \qquad b_1 \geq \mu \geq b_2, \qquad \beta_1 \geq \mu \geq \beta_2.$$

The path described by the moving body $M$ in the meridian plane, therefore, must always remain within one of the four trapezoidal regions of this plane bounded by ellipses $\lambda = a_1$, $\lambda = a_2$ and by the hyperbolas $\mu = b_1$, $\mu = b_2$, or by the hyperbolas $\mu = \beta_1$, $\mu = \beta_2$, as indicated in Fig. 2.

If, at the beginning of motion, $M$ is in one of these four regions, say I, it must always remain within this region. As the meridian plane describes the angle $\phi$, the region I will generate a tube-like region in space and within this space the motion of $M$ must take place. Whatever the nature of the trajectory within this space may be, it can never extend beyond it.

If $b_1 = \beta_2$, $b_2 = \beta_1$ numerically, then as the meridian plane rotates about its axis, the spaces generated by the regions I and II will be the same as those generated by IV and III respectively, but if $b_1 \neq \beta_2$, $b_2 \neq \beta_1$ numerically, this will not be the case. It may happen, however, that the spaces generated by I and IV, or II and III, overlap; then, supposing $M$ to be in I at the beginning of motion, the trajectory is limited by the combined spaces generated by I and IV after a semi-revolution of the meridian plane.

Let $a_0$ and $b_0$ be multiple roots of the equations $\Lambda = 0$ and $\mathrm{M} = 0$ respectively. Then, if at the beginning and throughout the motion we have $\lambda = a_0$, the motion will be upon an arc of the ellipse $\lambda = a_0$; and, as the meridian plane rotates about the $x$-axis, this ellipse will generate an ellipsoid, upon a certain portion of which the body $M$ must move. Likewise, if we have $\mu = b_0$ at the beginning and throughout the motion, the body $M$ will move upon an arc of the hyperbola $\mu = b_0$ in the meridian plane, and in space, upon a certain portion of the hyperboloid of revolution generated by the hyperbola branch $\mu = b_0$. If we have at the same time $\lambda = a_0$ and $\mu = b_0$, no motion will be possible in the meridian plane and the orbit in space will be a circle whose axis is the $x$-axis.

A further discussion of the conditions for the kinds of motion just mentioned will be given later.

When the motion is supposed to take place in a fixed plane passing through the five centres of force, that is, when $\alpha = 0$, we have

$$\begin{aligned} U = \frac{1}{\lambda^2 - \mu^2}\Big\{\Big(&-(m + \tfrac{1}{2} m_3 + m')\lambda^4 + (\tfrac{1}{2} m_3 + 2m')c^2\lambda^2 + (m_1 + m_2)\lambda \\ &+ 2m_4\sqrt{\lambda^2 - c^2} - \tfrac{1}{2}\cdot\frac{\beta c^2}{\lambda^2 - c^2}\Big) - \Big(-(m + \tfrac{1}{2} m_3 + m')\mu^4 + (\tfrac{1}{2} m_3 + 2m')c^2\mu^2 \\ &+ (m_1 - m_2)\mu - 2m_5\sqrt{c^2 - \mu^2} - \tfrac{1}{2}\cdot\frac{\beta c^2}{\mu^2 - c^2}\Big)\Big\}, \end{aligned}$$

$$\Lambda = f(\lambda) + (m_1 + m_2)\lambda^3 + 2m_4\sqrt{\lambda^2 - c^2},$$

$$\mathrm{M} = f(\mu) + (m_1 - m_2)\mu^3 - 2m_5\sqrt{c^2 - \mu^2},$$

$$f(s) = -(m + \tfrac{1}{2} m_3 + m')s^6 + (mc^2 + m_3c^2 + 2m'c^2 + h)s^4 + ks^2 + \tfrac{1}{2}\beta c^2 - kc^2.$$

When $\mu$ becomes zero, $U$ becomes infinite. When either $\lambda$ or $\mu$ becomes numerically equal to $c$, $U$ remains finite, but again becomes infinite when $\lambda = \mu = c$; that is, when there is a collision of $M$ with the real focus, $K_1$, or $K_2$. Hence the trajectory may cross the $y$-axis but not the $x$-axis, and it can not pass through the real foci $K_1$, $K_2$.

For sufficiently large values of $\lambda$, the function $\Lambda$ becomes and remains negative as $\lambda$ continues to increase. If we represent the greatest value of $\lambda$ for which $\Lambda$ changes from positive to negative values by $l$, then $\lambda$ must satisfy the inequality

$$+c \leq \lambda \leq l$$

in all cases of real motion.

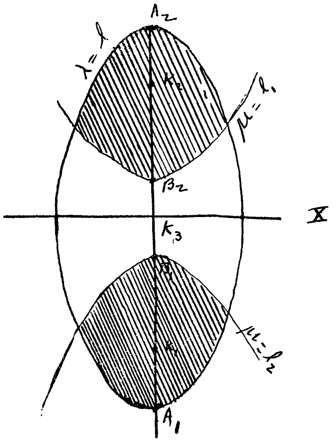

FIG. 3.

Since $U$ becomes infinite when $\mu$ becomes zero, $\mu$ can not approach zero indefinitely. Representing the lower limit of the positive values and the upper limit of the negative values which $\mu$ may take by $l_1$ and $l_2$ respectively, $\mu$ must always satisfy the inequalities

$$+c \geq \mu \geq l_1, \qquad l_2 \geq \mu \geq -c$$

in all cases of real motion.

From these limitations on $\lambda$ and $\mu$, it follows that the trajectory must lie wholly within one or the other of the two domains of the fixed plane, bounded by the ellipse $\lambda = l$ and the hyperbola branches $\mu = l_1$, $\mu = l_2$, as in Fig. 3.

For $\lambda = c$ the ellipse degenerates into the straight line segment $K_1 K_2$; for $\mu = c$ the hyperbola degenerates into the straight line $K_1 K_2$ beyond $K_2$, and for

$\mu = -c$ into the straight line $K_1 K_2$ beyond $K_1$. Hence, if at the beginning and throughout the motion we have $\lambda = c$, the body $M$ must move on the line $K_1 K_2$, either on the segment $B_2 K_2$ or the segment $B_1 K_1$, not including the limiting positions $K_2$, $K_1$. On the other hand, if at the beginning and throughout the motion we have $\mu = +c$, the body $M$ must move upon the line segment $K_2 A_2$, excluding $K_2$; and similarly for $\mu = -c$, the motion takes place on the straight line segment $K_1 A_1$, excluding $K_1$.

If we have at the beginning and throughout the motion $\lambda = a_0$, where $a_0$ is a multiple root of $\Lambda = 0$, the motion will be upon an arc of the ellipse $\lambda = a_0$. Likewise, if we have $\mu = b_0$ at the beginning and throughout the motion, the motion will be upon an arc of the hyperbola $\mu = b_0$.

For a fixed system of values of the masses involved, the existence of multiple roots of the equations $\Lambda = 0$, $\mathrm{M} = 0$ depends upon the arbitrary constants $h$ and $k$ that is, upon the initial conditions of the motion. From this it is evident that the same conic which can be described by all the forces acting at the same time can also be described when any one of the forces $R_1$, $R_2$, $R_3$ acts alone or when the two forces $R_4$, $R_5$ alone act. This fact Lagrange considered most remarkable and called attention to it in his work on the problem of two fixed centres in the *Mécanique Analytique.* Legendre* also noticed it. The above result and the theorems of Lagrange and Legendre are corollaries of the following general theorem due to Bonnet:†

Si plusieurs masses $m$, $m'$, $m''$, ...., respectivement soumises à l'action des forces $F$, $F'$, $F''$, ...., et partant toutes d'un point $A$ avec des vitesses $v_0, v_0', v_0''$, ...., de grandeur différente mais de même direction, decrivent la même courbe $ACB$, la masse quelconque $M$, soumise à l'action de la résultante des forces $F$, $F'$, $F''$, ...., et partant du point $A$ avec une vitesse $V_0$ ayant la même direction que les vitesses $v_0$, $v_0'$, $v_0''$, ...., décrira encore la courbe $ACB$, pourvu que les forces $F$, $F'$, $F''$, ...., indépendantes du temps et que la force vive initiale $MV$ de la masse $M$ soit égale à la somme

$$m v_0^2 + m' v_0'^2 + m'' v_0''^2 + \dots,$$

des forces vives initiales des masses $m$, $m'$, $m''$, .....

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* *Traité de fonctions elliptiques*, Vol. I, p. 426.

† *Journ. de Math.*, Vol. IX, pp. 113–115; *Œuvres de Lagrange*, Vol. XII, Note IV, pp. 353–355.

We shall now consider the motion of a particular case of the preceding problem by the method employed by Charlier* in the discussion of the motion of the original problem of two fixed centres when the motion is in a plane through the line of centres. The problem is as follows:

To determine the motion of a free body $M$ in space attracted toward two fixed centres $K_1$, $K_2$ by forces $R_1 = -m r_1 - \frac{m_1}{r_1^2}$, $R_2 = -m r_2 - \frac{m_2}{r_2^2}$ respectively, and by a third force $R_3 = -\frac{\beta}{x^3}$ parallel to the line $K_1 K_2$.

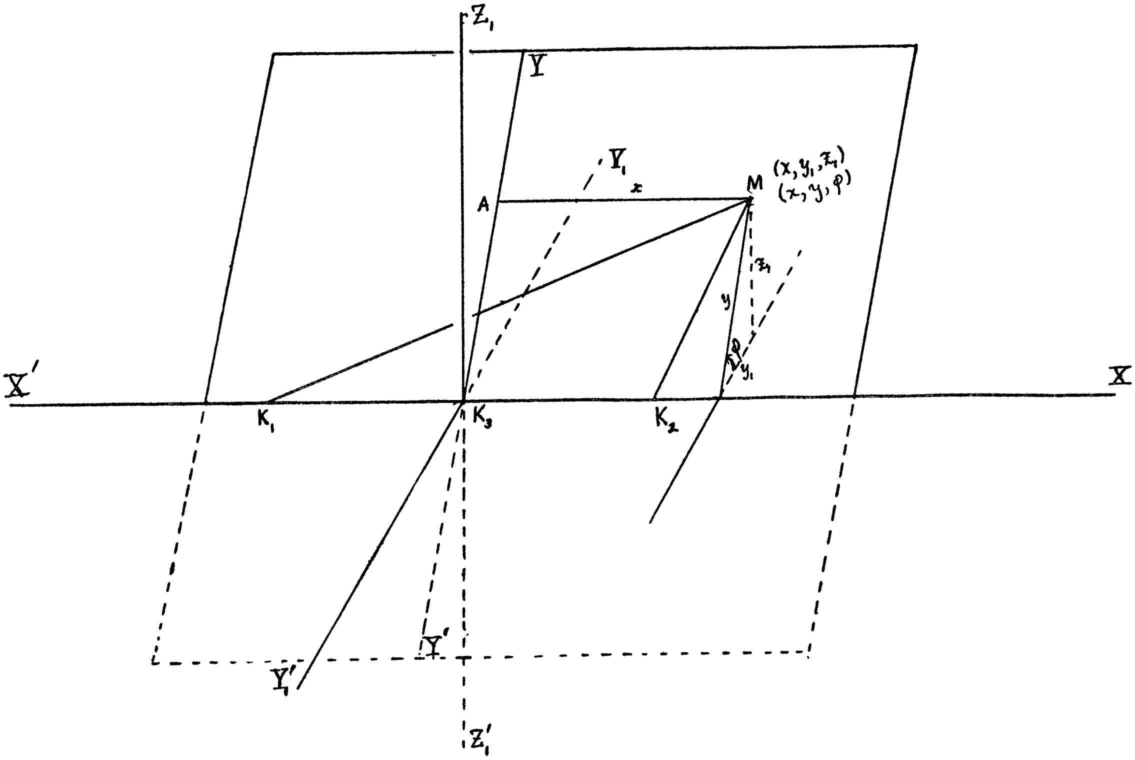

FIG. 4.

In this case $K_1$, $K_2$ are fixed. The plane $K_1 M K_2$ is supposed to rotate about the line $K_1 K_2$ as axis, coinciding with the fixed plane $X K_3 Y_1$ at the beginning of motion and describing the angle $\phi$ in the time $t$, as indicated in Fig. 4. $AM$, or $x$, is the distance of the moving body $M$ from the fixed plane $Y_1 K_3 Z_1$. Hence the third force, $R_3$, varies inversely as the cube of the distance of $M$ from the fixed plane $Y_1 K_3 Z_1$.

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* *Mechanik des Himmels*, Vol. I, pp. 117–163.

If we take the mass $M$ as the unit mass, the equations of motion are

$$\frac{d^2x}{dt^2} = -m(x+c) - m(x-c) - \frac{m_1(x+c)}{r_1^3} - \frac{m_2(x-c)}{r_2^3} - \frac{\beta}{x^3},$$

$$\frac{d^2y_1}{dt^2} = -my_1 - my_1 - \frac{m_1y_1}{r_1^3} - \frac{m_2y_1}{r_2^3},$$

$$\frac{dz_1}{dt^2} = -mz_1 - mz_1 - \frac{m_1z_1}{r_1^3} - \frac{m_2y_2}{r_2^3}.$$

The further details of the solution are altogether analogous to those of the general problem already solved, so that we content ourselves with the results

$$T = \tfrac{1}{2}(\lambda^2 - \mu^2)\left\{\frac{\lambda'^2}{\lambda^2 - c^2} - \frac{\mu'^2}{\mu^2 - c^2}\right\},$$

$$U = \frac{1}{\lambda^2 - \mu^2}\left\{\left(-m\lambda^4 + (m_1 + m_2)\lambda - \frac{\frac{1}{2}\alpha^2c^2}{\lambda^2 - c^2} - \frac{\frac{1}{2}\beta c^2}{\lambda^2}\right)\right.$$
$$\left. - \left(-m\mu^4 + (m_1 - m_2)\mu - \frac{\frac{1}{2}\alpha^2c^2}{\mu^2 - c^2} - \frac{\frac{1}{2}\beta c^2}{\mu^2}\right)\right\},$$

$$\frac{d\phi}{dt} = \frac{\alpha}{y^2},$$

$$W = \int \frac{d\lambda}{\lambda(\lambda^2 - c^2)}\sqrt{\Lambda} + \int \frac{d\mu}{\mu(\mu^2 - c^2)}\sqrt{\mathrm{M}},$$

$$\lambda(\lambda^2 - \mu^2)\frac{d\lambda}{dt} = \sqrt{\Lambda}, \tag{30}$$

$$\mu(\lambda^2 - \mu^2)\frac{d\mu}{dt} = -\sqrt{\mathrm{M}}, \tag{31}$$

$$dt = \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \frac{\mu^3 d\mu}{\sqrt{\mathrm{M}}}, \tag{32}$$

$$\frac{\lambda d\lambda}{\sqrt{\Lambda}} + \frac{\mu d\mu}{\sqrt{\mathrm{M}}} = 0, \tag{33}$$

$$d\phi = -\alpha c^2\left(\frac{\lambda d\lambda}{(\lambda^2 - c^2)\sqrt{\Lambda}} + \frac{\mu d\mu}{(\mu^2 - c^2)\sqrt{\mathrm{M}}}\right), \tag{34}$$

$$t + h_1 = \int \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \int \frac{\mu^3 d\mu}{\sqrt{\mathrm{M}}}, \tag{35}$$

$$k_1 = \int \frac{\lambda d\lambda}{\sqrt{\Lambda}} + \int \frac{\mu d\mu}{\sqrt{\mathrm{M}}}, \tag{36}$$

$$\phi + \alpha_1 = -\alpha c^2\left(\int \frac{\lambda d\lambda}{(\lambda^2 - c^2)\sqrt{\Lambda}} + \int \frac{\mu d\mu}{(\mu^2 - c^2)\sqrt{\mathrm{M}}}\right). \tag{37}$$

The integration constants are $h, k, \alpha, h_1, k_1, \alpha_1$. The symbols $\Lambda$ and M stand for the following expressions:

$$\Lambda = -2m\lambda^8 + 2(mc^2 + h)\lambda^6 + 2(m_1 + m_2)\lambda^5 - 2(hc^2 - k)\lambda^4$$
$$\quad - 2(m_1 + m_2)c^2\lambda^3 - (2k + \alpha^2 + \beta)c^2\lambda^2 + \beta c^4,$$
$$\mathrm{M} = -2m\mu^8 + 2(mc^2 + h)\mu^6 + 2(m_1 - m_2)\mu^5 - 2(hc^2 - k)\mu^4$$
$$\quad - 2(m_1 - m_2)c^2\mu^3 - (2k + \alpha^2 + \beta)c^2\mu^2 + \beta c^4;$$

or, setting for short

$$A = 2m, \quad B = 2mc^2 + h, \quad C_1 = 2(m_1 + m_2), \quad C_2 = 2(m_1 - m_2),$$
$$D = 2(hc^2 - k), \quad E_1 = 2(m_1 + m_2)c^2, \quad E_2 = 2(m_1 - m_2)c^2,$$
$$F = 2k + \alpha^2 + \beta, \quad G = \beta c^4$$
$$\Lambda = -A\lambda^8 + B\lambda^6 + C_1\lambda^5 - D\lambda^4 - E_1\lambda^3 - F\lambda^2 + G,$$
$$\mathrm{M} = -A\mu_8 + B\mu^6 + C_2\mu^5 - D\mu^4 - E_2\mu^3 - F\mu^2 + G.$$

It was shown by Serret* that under certain conditions

$$\Lambda\mathrm{M} = 0$$

is a singular solution of the problem of Euler or of that of Lagrange, and as his reasoning is directly applicable to the above problem we may conclude that this equation is also a singular solution of the problem in hand. In the original problem of Euler $\Lambda$ and M are of the fourth degree in $\lambda$ and $\mu$ respectively, in Lagrange's problem they are of the sixth, and in this problem they are of the eighth.

The condition for such singular solution is that $\lambda - \lambda_0$, $\lambda_0$ being the initial value of $\lambda$, be a multiple factor of $\Lambda$ of order $m$, $m \geq 2$; or similarly, that $\mu - \mu_0$, $\mu_0$ being the initial value of $\mu$, be a multiple factor of M of order $n$, $n \geq 2$; that is, $\lambda_0$ must be at least a double root of the equation $\Lambda = 0$; or $\mu_0$ at least a double root of $\mathrm{M} = 0$.

The trajectory in the meridian plane will therefore be the ellipse $\lambda = \lambda_0$, or the hyperbola $\mu = \mu_0$, according as

$$\Lambda_0 = 0, \quad \left(\frac{\partial\Lambda}{\partial\lambda}\right)_0 = 0,$$

or

$$\mathrm{M}_0 = 0, \quad \left(\frac{\partial\mathrm{M}}{\partial\mu}\right)_0 = 0,$$

where $\Lambda_0$ and $\left(\frac{\partial\Lambda}{\partial\lambda}\right)_0$ are the values of $\Lambda$ and its derivative with regard to $\lambda$

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**Œuvres de Lagrange*, Vol. XII, Note II, pp. 346–349.

when $\lambda = \lambda_0$, and likewise $\mathrm{M}_0$ and $\left(\frac{\partial \mathrm{M}}{\partial \mu}\right)_0$ are the values of M and its derivative with regard to $\mu$ when $\mu = \mu_0$.

As the meridian plane rotates about the line of centres, the ellipse $\lambda = \lambda_0$ will generate an ellipsoid of revolution and the hyperbola $\mu = \mu_0$ will generate an hyperboloid of revolution. Upon one of these surfaces the motion takes place.

But if $\lambda = \lambda_0$ and $\mu = \mu_0$ at the same time where $\lambda_0$ and $\mu_0$ occur at least twice as roots of the equations $\Lambda = 0$ and $\mathrm{M} = 0$ respectively, then no motion is possible in the meridian plane, for in this case

$$\frac{d\lambda}{dt} = 0, \qquad \frac{d\mu}{dt} = 0;$$

in space, however, the trajectory will be a circle whose axis is the axis of revolution and whose centre is $K_3$. This kind of motion was first remarked by Legendre,* and later by Andrade.†

With regard to equations of the form

$$\left(\frac{dx}{dt}\right)^2 = F(x)$$

Charlier ‡ established certain theorems which are immediately available for our discussion. The equation is supposed to be one capable of interpretation in a mechanical problem, and therefore $x$ as well as its first derivative with regard to $t$ may be regarded as real, finite, and continuous for all finite values of $t$.

In the first place let $a$ be any real root whatever of $F(x) = 0$ of order $m$, $m$ being a positive integer. Let $x = x_0$ at the beginning of motion and let $x_0$ be so taken that no zero value of $F(x)$ lies between $a$ and $x_0$. During the motion let $x$ approach $a$ continuously by increasing or decreasing values according as $x_0$ is less than or greater than $a$. Under these assumptions, Charlier succeeded in establishing the following results:

First, for $m \geq 2$, $x$ can not reach the value $a$ in a finite interval of time, and being real and continuous, it can not therefore pass the value $a$.

Second, for $m = 1$, $x$ will reach the value $a$ within a finite time; but it can not pass this value, for then the derivative will change sign, which is not admissible.

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* *Traité de fonctions elliptiques*, Vol. II, p. 529.

† *Journal d'École Polytechnique*, Cahier 60, 1890.

‡ *Mechanik des Himmels*, Vol. I, pp. 117–163.

In other words, $x$ can never pass a zero value of $F(x)$.

In the second place, let $a$ and $b$ be two consecutive roots of $F(x)=0$ of orders $m$ and $n$ respectively, let $a<b$ and let $x_0$ lie between $a$ and $b$ at the beginning of motion. Then it follows immediately from the above theorem that if $x=x_0$ at the beginning of motion, it must always remain within the interval $(a, b)$. If $m=n=1$ and if $x$ starts from $x_0$ toward $a$, it will continue to decrease until it reaches the value $a$; then it turns and continues to increase until it takes the value $b$, when it changes its direction again toward $a$, and so on indefinitely. The motion in this case is one of oscillation, or what Charlier calls *libration motion.* The limits $a$ and $b$ are called the *libration limits.*

If $m\geq 2$ and $n=1$, and if $x$ starts from $x_0$, $x_0$ being in the interval $(a, b)$ at the beginning of motion, toward $a$, it will continue to decrease but will never reach the value $a$; it approaches $a$ as a limit and can not attain this limit in a finite interval of time. If $x$ starts from $x_0$ toward $b$, it will reach $b$ in a finite interval of time; it will then change its direction and continue to decrease toward $a$ as before. If $m\geq 2$ and $n\geq 2$, then, in whichever direction $x$ moves at the beginning of motion, it will continue to move indefinitely, approaching either $a$ or $b$ as a limit. Charlier calls this kind of motion *limitation motion* and the limits, *limitation limits.*

These theorems will now be applied to equations (30) and (31).

The coordinates $\lambda$ and $\mu$ have been introduced by the equations

$$\lambda=\tfrac{1}{2}(r_1+r_2), \qquad \mu=\tfrac{1}{2}(r_1-r_2),$$

where $\lambda$ and $\mu$ are the parameters of a system of confocal conics having their real foci at the fixed centres of force, $K_1$, $K_2$. If we denote the distance $K_1 K_2$ by $2c$, then the least value of $\lambda$ is $+c$, while the greatest and least values of $\mu$ are $+c$ and $-c$ respectively. Since $\lambda$ is the semi-major axis of the ellipse, the semi-minor axis is $\sqrt{\lambda^2-c^2}$; of the hyperbola the semi-transverse axis is $\mu$, and the semi-conjugate axis is $\sqrt{c^2-\mu^2}$. Hence the equations of the ellipses and hyperbolas are respectively

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-c^2}=1, \qquad \frac{x^2}{\mu^2}-\frac{y^2}{c^2-\mu^2}=1,$$

and therefore

$$c^2x^2=\lambda^2\mu^2, \qquad c^2y^2=(\lambda^2-c^2)(c^2-\mu^2).$$

From these defining equations of $\lambda$ and $\mu$, it follows that $\lambda$ and $\mu$ must satisfy the inequalities

$$\infty > \lambda > c, \qquad +c \geq \mu \geq -c.$$

In order that the quantities under the radicals in the integrals may be positive, that is, in order that there may be real motion, $\lambda$ and $\mu$ must also satisfy the conditions

$$\Lambda \geq 0, \qquad \mathrm{M} \geq 0,$$

respectively. All of the above conditions $\lambda$ and $\mu$ must satisfy throughout the motion.

The potential function $U$ can be written

$$U = \frac{1}{2(\lambda^2 - \mu^2)} \left\{ \left( \frac{\Lambda}{\lambda^2(\lambda^2 - c^2)} - 2(h\lambda^2 + k) \right) - \left( \frac{\mathrm{M}}{\mu^2(\mu^2 - c^2)} - 2(h\mu^2 + k) \right) \right\}.$$

It is evident that, of the possible values $\lambda$ and $\mu$, $U$ becomes infinite for $\lambda = c$ and for $\mu = \pm c$ or zero. Therefore $\lambda$ and $\mu$ can not be allowed to take these values nor to approach them indefinitely, if we wish to avoid infinite discontinuities in $U$. Hence there must be a lower limit to the possible values of $\lambda$, as $l > c$; likewise, an upper and a lower limit to the possible positive values of $\mu$ which we denote by $l_1$ and $l_1'$ respectively, and an upper and lower limit to the possible negative values of $\mu$ which will be denoted by $l_2'$ and $l_2$ respectively. The following inequalities must therefore be fulfilled by $\lambda$ and $\mu$ respectively:

$$\infty > \lambda \geq l, \qquad l_1 \geq \mu \geq l_1', \qquad l_2' \geq \mu \geq l_2.$$

For any value of $\lambda > c$ for which $\Lambda$ vanishes, the function $U$ will be finite, however little this value of $\lambda$ may differ from $c$. Hence the least value of the possible values of $\lambda$ will always be less than or at most equal to the least value of $\lambda$ for which $\Lambda$ vanishes. In other words, all the positive roots of the equation $\Lambda = 0$ which are greater than $+c$ are also greater than or equal to $l$. Similarly, all the positive values of $\mu$ in the interval $(0, +c)$ for which M vanishes lie in the interval $(l_1', l_1)$, and all the negative values of $\mu$ in the interval $(-c, 0)$ for which M vanishes lie in the interval $(l_2, l_2')$.

When $\mu$ becomes zero, not only $U$ but also $d\mu/dt$ becomes infinite. But for this value of $\mu$, no discontinuities occur in the three final integrals of the solution; that is, the time $t$ as well as the trajectory in the meridian plane and in space is finite and continuous, as one can easily determine. Hence in a discussion of the trajectories no further attention need be given to these discontinuities in $U$ and $d\mu/dt$ and it will be sufficient that $\lambda$ and $\mu$ satisfy the conditions

$$\infty > \lambda \geq l, \qquad l_2 \geq \mu \geq l_1.$$

Without loss of generality we may take $m_1 > m_2$. Then, since $c^2$ and $\alpha^2$ are always positive, if we regard $m$, $m_1$, $m_2$ and $\beta$ as essentially positive, no change of sign can take place in any of the coefficients of $\Lambda$ and $\mathrm{M}$ which are independent of the arbitrary constants $h$ and $k$. Hence $A$, $C_1$, $C_2$, $E_1$, $E_2$, $G$ are all positive, while $B$, $D$, $F$ may be positive or negative, depending for their sign upon the values of $h$ and $k$.

When $\lambda = 0$, the expression $\Lambda$ is positive, and when $\lambda = \pm c$, $\Lambda$ is negative. $\Lambda$ must therefore have one real root between 0 and $+c$, and one between $-c$ and 0. By the aid of Descartes' Rule of Signs, we find also that $\Lambda$ can not have more than three positive, and it can not have more than five negative roots. $\Lambda = 0$ must therefore have at least one positive root, but may have three; and it must have at least one negative root, but may have three or five. The same is true relative to the roots of the equation $\mathrm{M} = 0$.

Further, for a certain value of $\lambda$, the function $\Lambda$ becomes negative and remains negative for all greater values of $\lambda$. Hence there must be a greatest value of $\lambda$ for which $\Lambda$ is positive or zero; that is, there must be a greatest value for which $\Lambda$ changes from positive to negative. This value, which we shall denote by $a_1$, must be a root of $\Lambda = 0$. Let us suppose $a_1 > c$. Since $\Lambda$ is negative for $\lambda = +c$ and positive for values of $\lambda$ less than $a_1$, $\Lambda$ must also change sign for some value of $\lambda$ between $+c$ and $a_1$; there must be a positive root of $\Lambda = 0$ in the interval $(+c, a_1)$. It is evident then, that $\Lambda$ is positive for all values of $\lambda$ in the interval $(a_2, a_1)$ and for no other values of $\lambda$ greater than $+c$. Since $a_1$ and $a_2$ are both greater than $l$, $\lambda$ may take all values in the interval $(a_2, a_1)$ including the limits $a_2$ and $a_1$.

We have seen that one positive root of $\Lambda = 0$ lies between 0 and $+c$. If $a_1$ be this root, then there can be but one positive real root; if $a_1$ be different from this root but less than $+c$, then $a_2$ will also be less than $+c$. In neither of these cases is motion possible. In all cases of real motion, therefore, we must have both $a_1$ and $a_2$ greater than $+c$, and $\lambda$ must satisfy the inequality $a_1 \geq \lambda \geq a_2$.

As to the roots of $\mathrm{M} = 0$ we have remarked that those only can have any influence upon the motion which satisfy the inequality $l_2 \geq \mu \geq l_1$, and that $\mathrm{M} = 0$ must have one positive root but may have three, and one negative root but may have three or five. We denote the positive roots by $b_1$, $b_2$, $b_3$ and the negative roots by $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, $\beta_5$. By $b_1$ we shall always mean the single positive root, or, if there be three positive roots, the greatest of the positive roots in the interval $(0, l_1)$ for which M changes from positive to negative values. Similarly, by $\beta_1$ we mean the negative root which must always exist, or, if there be more than one negative root, the greatest numerically of the negative roots in the interval $(l_2, 0)$ which causes M to change from negative to positive values for increasing values of $\mu$.

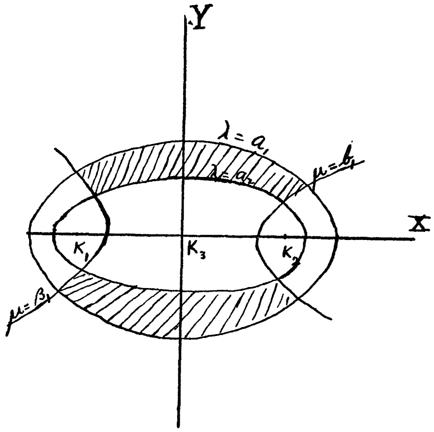

FIG. 5.

Hence it is clear that the trajectory of the moving body *M* in the meridian plane must remain wholly within one of two regions enclosed by $\lambda = a_1$, $\lambda = a_2$ and by the hyperbola branches $\mu = \beta_1$, $\mu = b_1$, as shown in Fig. 5. As the meridian plane rotates about the line of centres, these regions will generate a tube-like portion of space within which the trajectory must always remain.

Considering the definitions and restrictions with regard to the values of $\mu$ for which M vanishes as determined above, it is found that there are 105 different combinations of the roots of the equation $\mathrm{M} = 0$ which must be taken into account in a complete discussion of the motion, and each of these combinations must be associated with each of the two combinations of the roots of the equation $\Lambda = 0$, which are

$$A_1\colon\ a_1 > a_2, \qquad A_2\colon\ a_1 = a_2,$$

thus making in all 210 different combinations of the roots in question, which may influence the motion.

The positive roots of $\mathrm{M}=0$ may occur in any one of the following ways:

$B_1$: $b_2, b_3$ imaginary, or real and greater than $l_1$,
$B_2$: $b_2, b_3$ real, unequal and less than $b_1$,
$B_3$: $b_2, b_3$ real, equal and less than $b_1$,
$B_4$: $b_2, b_3$ real, equal, greater than $b_1$ and less than $l_1$,
$B_5$: $b_2, b_3$ real and equal to $b_1$.

The ways in which the negative roots may occur relative to $\beta_1$ and $l_1$ are accounted for in the following manner:

First, any two of them, as $\beta_2, \beta_3$:

$C_1$: $\beta_2, \beta_3$ imaginary, or real and less than $l_2$,
$C_2$: $\beta_2, \beta_3$ real, unequal and greater than $\beta_1$,
$C_3$: $\beta_2, \beta_3$ real, equal and greater than $\beta_1$,
$C_4$: $\beta_2, \beta_3$ real, equal, less than $\beta_1$ and greater than $l_2$,
$C_5$: $\beta_2, \beta_3$ real and equal to $\beta_1$.

Second, the remaining two, $\beta_4, \beta_5$:

$D_1$: $\beta_4, \beta_5$ imaginary, or real and less than $l_2$,
$D_2$: $\beta_4, \beta_5$ real, unequal and greater than $\beta_1$,
$D_3$: $\beta_4, \beta_5$ real, equal and greater than $\beta_1$,
$D_4$: $\beta_4, \beta_5$ real, equal, less than $\beta_1$ and greater than $l_2$,
$D_5$: $\beta_4, \beta_5$ real and equal to $\beta_1$.

We must also take certain of the $C$'s with certain of the $D$'s in order to obtain relations between the negative roots which are not given above. These are:

$C_3 D_{21}$: $\beta_4, \beta_5$ real, unequal, greater than $\beta_1$, and $\beta_2=\beta_3>\beta_5>\beta_4$,
$C_3 D_{22}$: $\beta_4, \beta_5$ real, unequal, greater than $\beta_1$, and $\beta_5>\beta_2=\beta_3>\beta_4$,
$C_3 D_{23}$: $\beta_4, \beta_5$ real, unequal, greater than $\beta_1$, and $\beta_5>\beta_4>\beta_3=\beta_2>\beta_1$,
$C_3 D_{24}$: $\beta_4, \beta_5$ real, unequal, greater than $\beta_1$, and $\beta_5=\beta_3=\beta_2>\beta_4$,
$C_3 D_{25}$: $\beta_4, \beta_5$ real, unequal, greater than $\beta_1$, and $\beta_5>\beta_4=\beta_3=\beta_2$,
$C_3 D_{31}$: $\beta_4, \beta_5$ real, equal, greater than $\beta_1$, and $\beta_2=\beta_3=\beta_4=\beta_5$,
$C_3 D_{32}$: $\beta_4, \beta_5$ real, equal, greater than $\beta_1$, and $\beta_5=\beta_4>\beta_3=\beta_2>\beta_1$,
$C_4 D_{41}$: $\beta_4, \beta_5$ real, equal, less than $\beta_1$, greater than $l_2$ and $\beta_5=\beta_4=\beta_3=\beta_2$,
$C_4 D_{42}$: $\beta_4, \beta_5$ real, equal, less than $\beta_1$, greater than $l_2$ and $\beta_2=\beta_3>\beta_4=\beta_5$.

All the combinations of the roots of $\Lambda = 0$ and $\mathrm{M} = 0$, which are actually different and which can have any influence on the motion, can now be formed. They are given by the following formulæ:

$$\begin{aligned} P_{ijk1} &= A_i B_j C_k D_1, & P_{ij3(2,k)} &= A_i B_j C_3 D_{2,k}, \\ P_{ijm2} &= A_i B_j C_m D_2, & P_{ijp(p,l)} &= A_i B_j C_p D_{p,l}, \\ P_{ijn3} &= A_i B_j C_n D_3, & P_{ij5n} &= A_i B_j C_5 D_n, \end{aligned}$$

where

$$\begin{aligned} &i = 1,\ 2; \quad j = 1,\ 2,\ 3,\ 4,\ 5; \quad k = 1,\ 2,\ 3,\ 4,\ 5; \\ &l = 1,\ 2; \quad m = 2,\ 4,\ 5; \quad n = 4,\ 5; \quad p = 3,\ 4. \end{aligned}$$

These different combinations, 210 in all, give rise to numerous possibilities of motion, depending upon the initial values of $\lambda$ and $\mu$; but, as one can easily determine, all the possible cases of motion so arising will fall into one of the following six general classes:

I. Libration in $\lambda$ and libration in $\mu$,
II. Libration in $\lambda$ and limitation in $\mu$,
III. Libration in $\lambda$ and $\mu =$ constant,
IV. $\lambda =$ constant and libration in $\mu$,
V. $\lambda =$ constant and limitation in $\mu$,
VI. $\lambda =$ constant and $\mu =$ constant.

To give a clear idea of the nature of the motion in the various possible cases it will be sufficient to discuss a few of them so as to illustrate each of the above six general classes. To this end we shall consider further only the combinations $P_{133(2,1)}$ and $P_{233(2,1)}$.

In the first of these combinations we have

$$a_1 > a_2, \qquad b_3 = b_2 < b_1, \qquad \beta_2 = \beta_3 > \beta_5 > \beta_4,$$

and therefore the following possibilities of motion are presented:

$$\begin{aligned} &(1) \quad a_1 > \lambda_0 > a_2: \quad b_1 > \mu_0 > b_2 = b_3; \\ &(2) \quad a_1 > \lambda_0 > a_2: \quad \mu_0 = b_2 = b_3; \\ &(3) \quad a_1 > \lambda_0 > a_2: \quad b_2 = b_3 > \mu_0 > \beta_2 = \beta_3; \\ &(4) \quad a_1 > \lambda_0 > a_2: \quad \mu_0 = \beta_2 = \beta_3; \\ &(5) \quad a_1 > \lambda_0 > a_2: \quad \beta_2 = \beta_3 > \mu_0 > \beta_5; \\ &(6) \quad a_1 > \lambda_0 > a_2: \quad \beta_4 > \mu_0 > \beta_1, \end{aligned}$$

$\lambda_0$ and $\mu_0$ being the initial values of $\lambda$ and $\mu$ respectively. In each of the six cases there is libration in $\lambda$. In (6) there is also libration in $\mu$ and therefore (6) belongs to class I. In (1), (3), (5) there is limitation in $\mu$ as in class II, and (2) and (4) belong to class III.

If at the beginning of the motion the conditions (6) are satisfied, the trajectory of the moving point in the meridian plane must always remain within one of the two trapezoidal regions bounded by the ellipses $\lambda = a_1$, $\lambda = a_2$ and by the hyperbola branches $\mu = \beta_4$, $\mu = \beta_1$. Since there is libration in both $\lambda$ and $\mu$, these variables will take the values $a_1$, $a_2$ and $\beta_4$, $\beta_1$ respectively an indefinite number of times and the trajectory will meet the bounding arcs an indefinite number of times. As the meridian plane rotates about its axis, this region of the meridian plane will generate a tube-like portion of space, bounded by the surfaces of the ellipsoids of revolution $\lambda = a_1$, $\lambda = a_2$ and the hyperboloids of revolution $\mu = \beta_4$, $\mu = \beta_1$. Within this tube the trajectory must always remain, meeting the limiting surfaces an indefinite number of times. The trajectory will be a coil-like curve.

If conditions (1) are satisfied at the beginning of motion, the trajectory in the meridian plane must always remain within the region bounded by the ellipses $\lambda = a_1$, $\lambda = a_2$ and the hyperbolas $\mu = b_1$, $\mu = b_2$. The variable $\lambda$ will take the values $a_1$, $a_2$ an indefinite number of times while $\mu$ may take the value $b_1$ once, if at all, and must then approach $b_2$ as a limit, not reaching it in a finite interval of time. Hence the trajectory will meet the bounding arcs of the ellipses an indefinite number of times and approach the bounding arc of the hyperbola $\mu = b_2$ in an asymptotic manner. In space the trajectory will be a wave-like, spiral-like curve meeting the surfaces of the ellipsoids of revolution $\lambda = a_1$, $\lambda = a_2$ an indefinite number of times while it continues to approach asymptotically the surface of the hyperboloid of revolution $\mu = b_2$, which is intercepted between the ellipsoids $\lambda = a_1$, $\lambda = a_2$.

When conditions (3) or (5) hold at the beginning of motion, the trajectory will be of the same general character as that corresponding to conditions (1).

When (2) and (4) are satisfied at the beginning of motion, we have libration in $\lambda$ and $\mu =$ constant throughout the motion. In the meridian plane the trajectory consists of the arc of the hyperbola intercepted between the ellipses $\lambda = a_1$, $\lambda = a_2$ and the motion is one of oscillation upon this arc. In space the trajectory must remain upon the surface of the hyperboloid $\mu = b_2$ intercepted between the ellipsoids generated by $\lambda = a_1$, $\lambda = a_2$. It is therefore a wave-like curve crossing this surface and meeting the bounding circles an indefinite number of times.

In the second combination selected for further discussion, $P_{233(2,1)}$, we have

$$a_1 = a_2, \qquad b_3 = b_2 < b_1, \qquad \beta_2 = \beta_3 > \beta_5 > \beta_4 > \beta_1,$$

and therefore we have the following possibilities depending upon the initial values $\lambda_0$ and $\mu_0$ of $\lambda$ and $\mu$ respectively:

(1) $\lambda_0 = a_1 = a_2$: $b_1 > \mu_0 > b_2 = b_3$;

(2) $\lambda_0 = a_1 = a_2$: $\mu_0 = b_2 = b_3$;

(3) $\lambda_0 = a_1 = a_2$: $b_2 = b_3 > \mu_0 > \beta_2 = \beta_3$;

(4) $\lambda_0 = a_1 = a_2$: $\mu_0 = \beta_2 = \beta_3$;

(5) $\lambda_0 = a_1 = a_2$: $\beta_2 = \beta_3 > \mu_0 > \beta_5$;

(6) $\lambda_0 = a_1 = a_2$: $\beta_5 > \mu_0 > \beta_4$.

Of these different cases (6) belongs to class IV; (1), (3), (5) to class V and (2), (4) to class VI.

If conditions (6) are satisfied at the beginning of motion, the trajectory will consist of the arc of the ellipse $\lambda = a_1$, between the hyperbolas $\mu = \beta_5$, $\mu = \beta_4$. In space it will be upon the surface of the ellipsoid of revolution bounded by the circles of intersection of this surface with the two hyperboloids of revolution $\mu = \beta_5$, $\mu = \beta_4$. The trajectory will be a wave-like curve upon the ellipsoid crossing its surface and meeting the bounding circles an indefinite number of times.

If at the beginning of motion conditions (1) are fulfilled, the trajectory in the meridian plane will be the arc of the ellipse $\lambda = a_1$ intercepted between the hyperbolas $\mu = b_1$, $\mu = b_2$, but the motion will not be one of oscillation as in the preceding case. If the variable $\mu$ move initially in the direction of $b_1$, it will reach that value in a finite time, then change its direction toward $b_2$, which it approaches as a limit, not reaching it in a finite interval of time. As the meridian plane rotates about its axis, the motion will be upon the surface of the ellipsoid of revolution $\lambda = a_1$, between the hyperboloids of revolution $\mu = b_1$, $\mu = b_2$. The trajectory will be a curve winding about the ellipsoid and approaching asymptotically the circle of intersection of this surface with the hyperboloid $\mu = b_2$. In cases (3) and (5) the trajectory is of the same nature as that just described.

In cases (2) and (4) no motion is possible in the rotating plane, but in space the trajectory is a circle whose axis is the line $K_1 K_2$. In (2), for example, the trajectory is the circle of intersection of the ellipsoid of revolution $\lambda = a_1$ and the hyperboloid $\mu = b_2$.

By giving particular values to the masses and the arbitrary constants, interesting particular cases of motion may be discovered. If we take $\alpha = 0$ and make the necessary changes throughout, the cases of motion in a fixed plane passing through the line of centres will be obtained. Then, if we take $\lambda = c$ throughout the motion, the trajectory will consist of the straight line segment $K_1 K_2$; or, if we take $\mu = \pm c$ throughout the motion, the trajectory will consist of the segments of the line $K_1 K_2$ beyond the fixed centres $K_2$ and $K_1$ respectively.

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